Introduction to Measurement Error

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- Why does it matter?
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  - measurement error can also bias causal inference

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- There are ways to anticipate its impact
  - and to some extent adjust for it
  - but to do so we first need to define these errors formally

Defining Measurement Error Formally

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- The classical measurement error model (random errors)
  - $\overbrace{X^*}^{observed} = \overbrace{X}^{true\ value} + \overbrace{U}^{noise}$
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- Only the variance is affected
  - $\sigma^2_{X^*} = \sigma^2_X + \sigma^2_U$; but the mean is unaffected since $E(U) = 0$
  - taking repeated observations we can estimate the prevalence of classical measurement error
  - the reliability ratio: $\rho_{X^*} = \dfrac{\sigma^2_X}{\sigma^2_X + \sigma^2_U} = \dfrac{\text{true variability}}{\text{observed variability}}$

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- Repeated observations won't detect systematic errors
  - we need a *gold standard*
  - e.g. unemployment register, victimisation surveys

Multiplicative Errors

- What if the error is proportional to the true value of the quantity being measured?
  - e.g. memory failures, count data

    How many alcohol units do you drink per week?

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- These can be better specified using a multiplicative rather than an additive model
  - $X^* = X \cdot U$, rather than $X^* = X + U$,

    with $E(U) = 1$ if random, and $E(U) \neq 1$ if systematic

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- Differential errors (correlated with the cause or the effect)

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  - random errors affect measures of dispersion, systematic errors affect measures of centrality
- But how does measurement error affect estimates from multivariate (regression) models?

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- Question: Is β biased in any of those scenarios?

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    $Y = \alpha + \beta X + (\epsilon - U)$
  - the measurement error is absorbed by the model's error term, affecting precision, but leaving regression coefficients unbiased

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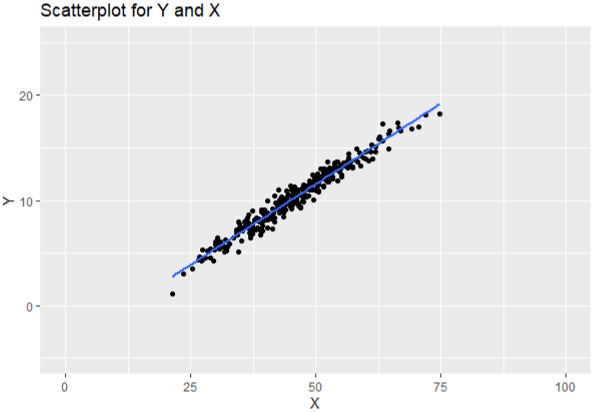

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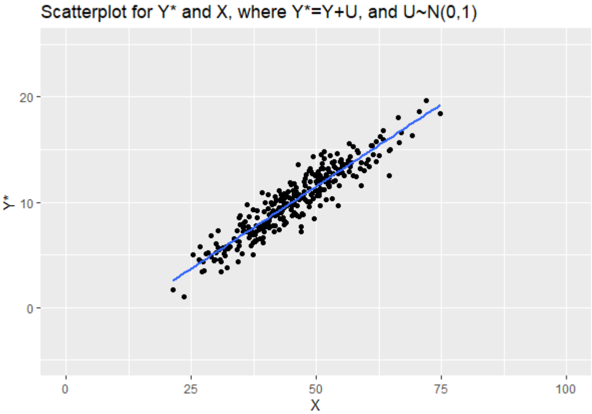

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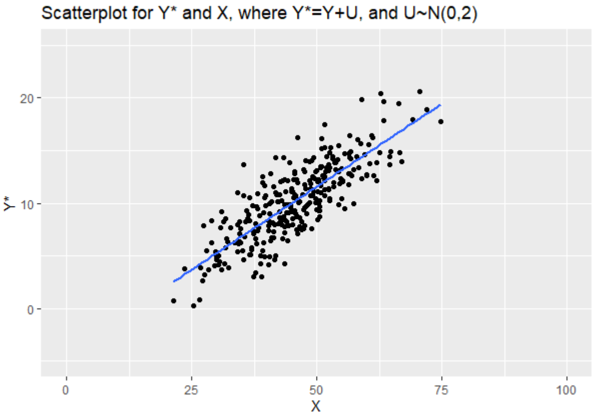

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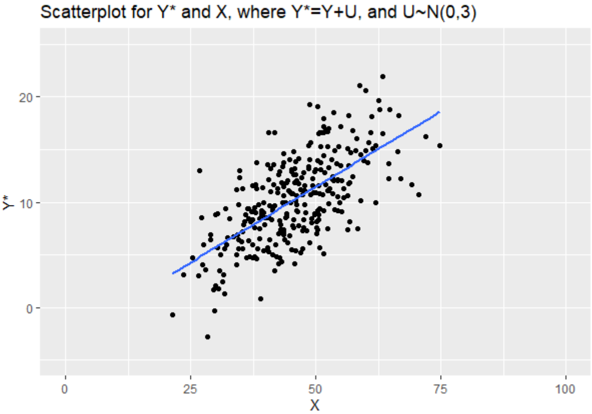

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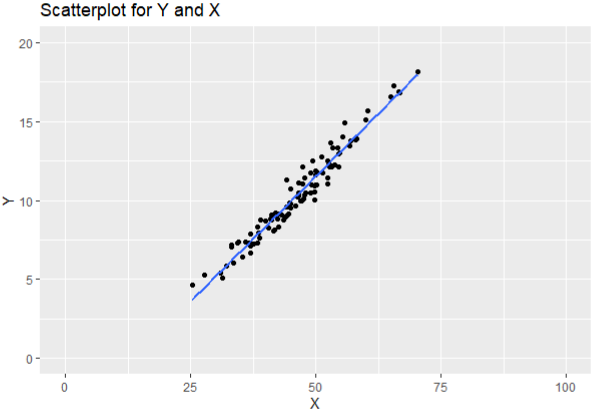
Using the properties of the covariance and the variance we can see how the former is not affected by random noise, but the latter is:

$$\sigma_{X^*,Y} = \sigma_{X+U,Y} = \sigma_{X,Y} + \sigma_{U,Y} = \sigma_{X,Y}$$
$$\sigma_{X^*}^2 = \sigma_{X+U}^2 = \sigma_X^2 + \sigma_U^2 + \sigma_{X,U} = \sigma_X^2 + \sigma_U^2$$

  - We can see this effect using simulated data

Effect of Random Measurement Error



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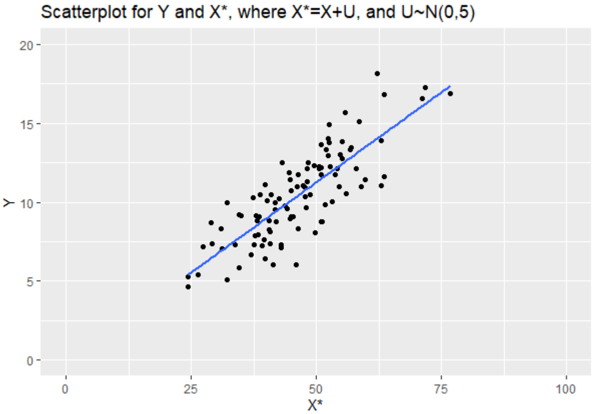

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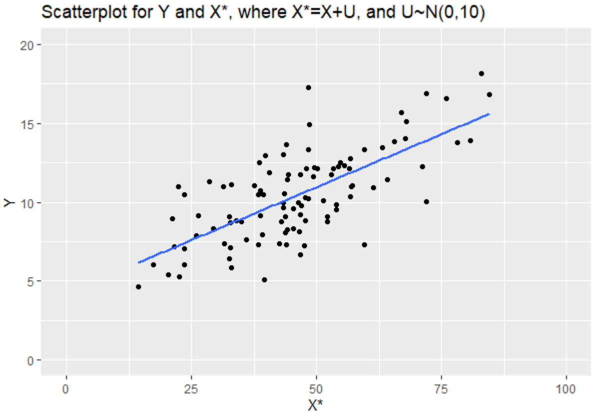

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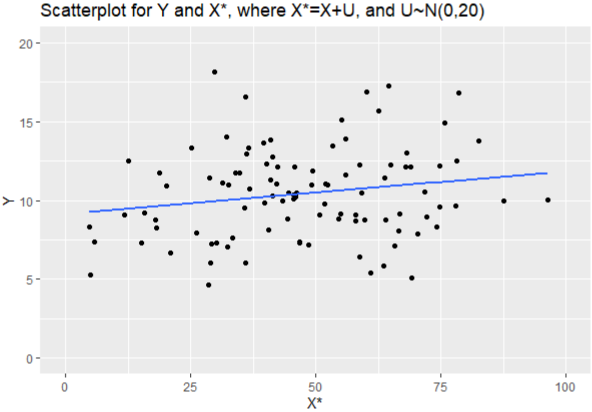

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    $Y \cdot U = \alpha + \beta X + \epsilon$

    $$Y = \frac{\alpha + \beta X + \epsilon}{U}$$
  - all regression coefficients are biased

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